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r-MAXIMAL MAJOR SUBSETS

BY

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ABSTRACT

The question of which r.e. sets A possess major subsets B which are also r-maximal in A ($A \subset_{rm} B$) arose in attempts to extend Lachlan's decision procedure for the $\forall\exists$-theory of $\mathscr{E}^*$, the lattice of r.e. sets modulo finite sets, and Soare's theorem that A and B are automorphic if their lattice of supersets $\mathscr{L}^*(A)$ and $\mathscr{L}^*(B)$ are isomorphic finite Boolean algebras. We characterize the r.e. sets A with some $B \subset_{rm} A$ as those with a Δ_3 function that for each recursive R_i specifies R_i or $\bar{R}_i$ as infinite on $\bar{A}$ and to be preferred in the construction of B. There are r.e. A and B with $\mathscr{L}^*(A)$ and $\mathscr{L}^*(B)$ isomorphic to the atomless Boolean algebra such that A has an rm subset and B does not. Thus $\langle \mathscr{E}^*, A\rangle$ and $\langle \mathscr{E}^*, B\rangle$ are not even elementarily equivalent. In every non-zero r.e. degree there are r.e. sets with and without rm subsets. However the class $\boldsymbol{F}$ of degrees of simple sets with no rm subsets satisfies $\boldsymbol{H}_1 \subsetneq \boldsymbol{F} \subsetneq \bar{\boldsymbol{L}}_2$.

Introduction

This paper grew out of a meeting of two different lines of investigation into the structure of $\mathscr{E}^*$, the lattice of recursively enumerable sets modulo finite sets. The first was an attempt to extend Lachlan's decision procedure for the $\forall\exists$-theory of $\mathscr{E}^*$ [2] by adding on a predicate to distinguish maximal sets. (This should be viewed as a first step towards a decision procedure for higher quantifier levels since they can be reduced to the $\forall\exists$-level by adding on the appropriate predicates. This approach is being attempted by Lerman and Soare [8].)

As in Lachlan's procedure one begins by trying to rule out as many sentences as possible by considering certain "canonical" configurations of r.e. sets. It turns out that an important new ingredient involves deciding whether or not certain simple sets, in particular whether hyperhypersimple (hhs) sets, have major subsets which are also r-maximal in them. (*Warning*: All sets and degrees named in this paper will be r.e.) With this convention in mind we define the following notions: B is a *major subset* of A, written $B \subset_m A$, iff $A - B$ is infinite and

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$$(\forall W)[A \cup W =^* \mathbf{N} \Rightarrow B \cup W =^* \mathbf{N}]$$

where $A \subseteq^* B$ denotes that $A - B$ is finite and $A =^* B$ that $A \subseteq^* B$ and $B \subseteq^* A$. We say B is *r-maximal in* A, written $B \subset_r A$, iff $A - B$ is infinite and r-cohesive i.e.

$$(\forall \text{ recursive } R)[A - B \subseteq^* R \vee A - B \subseteq^* \bar{R}].$$

Now it is easy to see that some sets A (even hhs sets) have major subsets B which are r-maximal in A (written $B \subset_{rm} A$). Thus for example if A is r-maximal (in $\mathbf{N}$) then any $B \subset_m A$ is also r-maximal in A: For any recursive R we have that $\bar{A} \subseteq^* R$ or $\bar{A} \subseteq^* \bar{R}$ by the r-maximality of A. Now $B \subset_m A$ implies that $\bar{B} \subseteq^* R$ or $\bar{B} \subseteq^* \bar{R}$ respectively, so R cannot split $A - B$. The difficulty arises in trying to see if various other simple sets also have rm subsets. Indeed a new sentence in this class which must be decided is

$$(\exists A)[A \text{ is hhs } \& \ A \text{ has no maximal superset } \& \ (\exists B)[B \subset_m A]].$$

It turns out that this property depends on the existence of certain Δ_3 functions describing some of the behavior of the recursive sets on $\bar{A}$. Somewhat surprisingly this leads to the connection with the second line of investigation — automorphisms of $\mathscr{E}^*$.

Our starting point on this line was Soare's theorem [17] that if M_1 and M_2 are maximal sets they are automorphic, i.e. there is an automorphism φ of $\mathscr{E}^*$ such that $\varphi(M_1) = M_2$. One way of viewing this result is as saying that if the lattice of supersets of M_1 and M_2 modulo finite sets are isomorphic to the trivial Boolean algebra $\{0, 1\}$ ($\mathscr{L}^*(M_1) \cong \mathscr{L}^*(M_2) \cong \mathbf{2}$) then M_1 and M_2 are in fact automorphic. Indeed when it is viewed this way Soare also shows in [17] that the result easily extends from $\mathbf{2}$ to any finite Boolean algebra. The natural question at that point is whether the result can be extended any further: In particular whether it also works for the countable atomless Boolean algebra which is perhaps the simplest infinite one. (As $\mathscr{L}^*(A)$ being a Boolean algebra is equivalent to A being hhs by [1, theorem 3] we call such sets *atomless* hhs sets.)

Now all automorphisms of $\mathscr{E}^*$ that have so far been explicitly constructed to give new elements automorphic to a given set have been given effectively on at least a reasonable class of representatives for the r.e. sets. (What is called a *skeleton* in [17].) This turns out to be equivalent to constructing a Δ_3 (on indices) automorphism of $\mathscr{E}^*$. Thus a reasonable starting point for trying to extend Soare's theorem was whether all atomless hhs sets have Δ_3 isomorphisms between their lattices of supersets. We shall note that Lachlan's construction [1, theorem 6] of an $\mathscr{L}^*(A)$ isomorphic to a given Σ_3 Boolean algebra actually produces a Δ_3 map. Thus by constructing arbitrary Σ_3 atomless Boolean algebras which are not Δ_3 isomorphic one can produce atomless hhs sets with no Δ_3 isomorphism between their lattices of supersets. More importantly one can control the Δ_3 properties of these Boolean algebras enough to either guarantee or rule out the existence of the type of Δ_3 maps associated with rm subsets. As the isomorphism to the $\mathcal{L}^*(A)$ constructed is Δ_3 the desired properties carry over and one constructs atomless hhs sets with and without rm subsets. Of course this answers our question on automorphisms negatively since these sets cannot be automorphic.

The organizational plan of this paper is therefore as follows: We study rm subsets in §1 and develop necessary and sufficient conditions for sets to have such subsets. In the second section we construct the various Σ_3 Boolean algebras needed and analyze Lachlan's construction of the associated hhs sets A_i enough to show that the desired properties of these algebras are carried over to $\mathcal{L}^*(A_i)$. Next in §3 we consider the problem of which degrees contain sets with or without rm subsets. Here simplicity will play an unusual role: Every simple set A in $\boldsymbol{L}_2$ (i.e. $A'' \equiv_T \emptyset''$) has an rm subset but there are sets with no rm subset in every nonzero degree. We conclude with some remarks and open questions in §4. Our notation is fairly standard and we cite Rogers [12] as a reference for basic facts and notation.

1. r-maximal major subsets

The question of whether various simple sets have rm subsets arose as we said in developing an extension of Lachlan's decision procedure for the $\forall\exists$-theory of $\mathcal{E}^*$ in the language of Lachlan [2] plus a predicate $M(X)$ to be interpreted as "X is a maximal set". Although as far as this problem is concerned only simple sets are relevant the more general question of which sets have rm subsets also arises in considering another of Lachlan's results. In response to a question of Rogers, Lachlan [1, p. 32] showed that no set A has a major subset B which is also maximal (or even hhs) in A. Thus if $B \subset_m A$, $A - B$ cannot be too thin (i.e hh-immune [1, p. 32]).

The next question along these lines of $A - B$ being "thin" is whether we can have $B \subset_m A$ and simultaneously B r-maximal in A (i.e. $B \subset_{rm} A$). If A is r-maximal, this is clearly possible. Although there are other examples it is not always possible to find $B \subset_{rm} A$. We characterize exactly when such a B exists in terms of a certain Δ_3 condition on A.

Let us begin by considering the difficulties encountered in trying to construct an rm subset B of a given set A. The construction is to be a movable marker one with the markers resting on the (potential) elements of $A - B$. To guarantee that $B \subset_m A$ as in [1, theorem 7] one would try to maximize the e-states of the markers with respect to the W_i, $i < e$, which so far appear to include $\bar{A}$ (so that $W_i \cup A = \mathbf{N}$ is threatened). (Of course some argument is needed later to show that these requirements guarantee that $B \subset_m A$. However, this is all that is actually done in the construction itself.) Note also that, as we shall see later, a reduction argument shows that it suffices to consider only recursive sets in this list. On the other hand to make $B \subset_r A$ we must for each recursive set R_i guarantee that $A - B \subseteq^* R_i$ or $A - B \subseteq^* \bar{R}_i$. The natural attempt here is to try to maximize the e-state of the markers with respect to some effective list $\{R_i : i \in \omega\}$ of the recursive sets. Unfortunately these requirements as presented conflict rather severely. Thus for example if maximizing the state of the markers with respect to R_0 has highest priority and $R_0 \subseteq A$ then we will never satisfy the majoricity requirement for $\bar{R}_0$ ($= W_0$ say). Instead we will have $\bar{R}_0 \cup A = \mathbf{N}$ but $A - B \subseteq R_0$ so that $B \not\subset_m A$. Now this does not mean that A has no rm subset for we could satisfy the r-cohesiveness requirement just as well by making all the markers move to $\bar{R}_0$ instead of R_0. There would then be no conflict whatsoever with the majoricity requirements for W_0. Of course we could have this problem with every R_i. What we seem to need is a way of deciding which side of the r-cohesiveness requirement (R_i or $\bar{R}_i$) to prefer when trying to satisfy this requirement. At the very least we must choose a side which intersects $\bar{A}$ infinitely often. Moreover if we expect to succeed on this side (e.g., $A - B \subseteq^* R_i$) it must be infinite on A as well. One more moment's thought shows that this is not sufficient. Our choices must cohere. Thus we cannot prefer R_0 and also $\bar{R}_1$ if $R_0 \cap \bar{R}_1 \cap \bar{A}$ is finite even though both $R_0 \cap \bar{A}$ and $\bar{R}_1 \cap \bar{A}$ are infinite. We are therefore led to the following notion:

Definition 1.1. Let $R_i = \{x \mid (\forall y \leqq x)[\varphi_i(y) \text{ convergent } \& \ \varphi_i(x) = 1]\}$. We say that a $\{0, 1\}$ valued function h is a *preference function for A* if for every initial segment σ of h ($\sigma \subset h$)

(1) $R_\sigma \cap \bar{A}$ is infinite and

(2) $R_\sigma \cap A$ is infinite, where $R_\sigma = \bigcap_{i \leqq \mathscr{L}\mathrm{th}\,\sigma} R^{\sigma(i)}$, $R_i^1 = R$ and $R_i^0 = \bar{R}_i$.

Note that even for simple sets A these conditions seem stronger than the ones we originally thought sufficient [6]. One simplification does however appear if A is simple: Condition (2) is automatically satisfied. Thus for simple sets we need only verify that a function h satisfies (1). One easy consequence of this is the following:

1.1(a) If h is a preference function for A and $B \subset A$ is simple then h is also a preference function for B.

Our next theorem will then imply that if A has an rm subset and $B \subset A$ is simple then B has an rm subset also. For non-simple sets however having an rm subset will not be inherited downward.

Now even a preference function for A will be of no use if it is too complex as we must enumerate B effectively. Although we cannot hope for a recursive one this much effectiveness is not required. We can get by in the construction with a Δ_3 preference function via suitable approximations. Moreover it turns out that the existence of a Δ_3 preference function for A is also a necessary condition for the existence of an rm subset of A.

THEOREM 1.2. *A has an* rm *subset iff it has a* Δ_3 *preference function.*

PROOF. Suppose first that $B \subset_{rm} A$. We define the required function by $h(i) = 1 \Leftrightarrow A - B \subseteq^* R_i$ ($\Leftrightarrow (\exists x)(\forall y > x)$ $[y \in A - B \rightarrow y \in R_i]$) and $h(i) = 0 \Leftrightarrow A - B \subseteq^* \bar{R}_i$ ($\Leftrightarrow (\exists x)(\forall y > x)$ $[y \in A - B \rightarrow y \notin R_i]$). As $A - B$ is r-cohesive exactly one of these Σ_3 alternatives must occur so that h is well defined on $\mathbf{N}$ and Δ_3. By definition $A - B \subseteq^* R_\sigma$ for every $\sigma \subseteq h$ and so $R_\sigma \cap A$ is certainly infinite. If $R_\sigma \cap \bar{A}$ were finite then $\bar{R}_\sigma \cup A =^* \mathbf{N}$. On the other hand $A - B \subseteq^* R_\sigma$ so that $\bar{R}_\sigma \cup B \neq^* \mathbf{N}$. This would then contradict the majority of B in A. Thus h is the desired Δ_3 preference function for A.

Next suppose that h is a Δ_3 preference function for A. As h is Δ_3 there is, by the well known limit lemma [14, p. 29], a recursive function $f(i, n, s)$ such that, for every i, $\lim_{n\to\infty} \lim_{s\to\infty} f(i, n, s) = h(i)$. We will use this function to approximate h in our construction. We will also approximate R_i^1 and R_i^0:

$$x \in R_{i,s}^1 \Leftrightarrow (\forall y \leqq x)[\varphi_{i,s}(y) \text{ is convergent}]$$

$$\& \ \varphi_{i,s}(x) = 1$$

$$x \in R_{i,s}^0 \Leftrightarrow (\forall y \leqq x)[\varphi_{i,s}(y) \text{ is convergent}]$$

$$\& \ \varphi_{i,s}(x) = 0.$$

($\varphi_{i,s}(y)$ means do s steps in the computation of $\varphi_i(y)$.) Note that while $\bigcup_{i<\omega} R_{i,s}^1 = R_i^1$ in every case, $\bigcup_{i<\omega} R_{i,s}^0 = R_i^0$ only if R_i is infinite. This will not bother us in the construction since if R_i is finite the cohesiveness requirement for it is automatically satisfied.

Our construction of B is by the standard movable marker type of argument. We enumerate A in stages via a 1–1 recursive function $a(s)$ and put $A_s =$

$\{a(i) \mid i < s\}$. We have markers Λ_n resting on various elements in A but not yet in B, say Λ_n^s, at stage s. They are moved to maximize their n-states with respect to our approximation to the sequence $\{R_i^{h(i)}\}$. To be precise we define the n-state of x at stage s to be the map $\sigma(n,x)$ from $n+1$ into $\{0,1\}$ given by

$$\sigma(n,x)(i) = \begin{cases} 1 & \text{if } x \in R_{i,s}^{f(i,n,s)}, \\ 0 & \text{otherwise.} \end{cases}$$

The n-states are ordered lexicographically and in moving the markers Λ_n to elements in higher n-states priority is given to lower numbers. At the conclusion of each stage elements in A_s without a marker are enumerated in B. Finally at the end of the construction we argue that the markers come to rest so that $A - B$ is infinite and that the maximization of n-states does guarantee that $B \subset_{rm} A$.

CONSTRUCTION. Stage s: Put the least marker Λ_k not on any number on $a(s)$ so $\Lambda_k^s = a(s)$. Let $\langle n, m \rangle$ be the (lexicographically) least pair such that $n < m$ and $\sigma \subsetneq h$ (If none exists go on to stage $s+1$.) Now set $\Lambda_i^{s+1} = \Lambda_i^s$ for $i < n$, $\Lambda_n^{s+1} = \Lambda_m^s$ and remove the Λ_i for $i > n$. All numbers that have been enumerated in A but are not equal to some Λ_i^{s+1} are now enumerated in B.

LEMMA 1.3. *$\mathrm{Lim}_{s\to\infty} \Lambda_n^s = \Lambda_n^\omega$ exists for every n.*

PROOF. By induction assume that $\Lambda_i^s = \Lambda_i^\omega$ for every $i < n$. Let s_0 be large enough so that $(\forall i \leqq n)(\forall s > s_0)$ $(f(i,n,s) = f(i,n,s_0))$. Thus at every stage $s > s_0$ Λ_n is trying to move to a number of maximum n-state with respect to the same fixed sequence of sets $\{R_i^{f(i,n,s_0)}\}_{i \leqq n}$. Once this sequence is fixed the n-state at stage s of any number is as usual a non-decreasing function of s. As there are only a finite number of n-states Λ_n can move to an element in a higher n-state only finitely often after stage s_0. Moreover, by the induction assumption, Λ_n^s can never be changed for the sake of increasing the i-state of any Λ_i^s $i < n$. Thus Λ_n^s changes only finitely often after stage s_0 and so is eventually constant. □

LEMMA 1.4. *$(\forall i)(A - B \subseteq^* R_i^{h(i)})$.*

PROOF. Say i_0 is the least counterexample. Let σ_0 be h restricted to i_0 and σ_1 be h restricted to i_0+1. Choose n_0 large enough so that

$$(\forall i \leqq i_0)(\forall n \geqq n_0)\left(\lim_{s\to\infty} f(i,n,s) = h(i)\right).$$

By assumption we may choose $n_1 > n_0$ so that $\Lambda_{n_1}^\omega \notin R_{i_0}^{h(i_0)}$ but $\Lambda_{n_1}^\omega \in R_{\sigma_0}$. Moreover we may assume that $\Lambda_{n_1}^\omega > \max\{x \mid x \in R_i \;\&\; i < i_0 \;\&\; R_i \text{ is finite}\}$. We now choose s_0 large enough so that $\Lambda_{n_1}^\omega = \Lambda_{n_1}^{s_0}$ and $(\forall i \leqq i_0)(\forall s \geqq s_0)$

$(f(i, n_1, s) = h(i))$. We also assume that if $s > s_0$ then $a(s) > \max\{x \mid x \in R_i$ & $i < i_0$ and R_i finite$\}$. For our contradiction we now consider the possibility of an $s > s_0$ such that $(\forall i \leqq i_0)(R_i \text{ infinite} \rightarrow a(s) \in R_{i,s}^{f(i,n_1,s)})$. If there is such an s then, at stage s, $\Lambda_{n_1}^s$ would be changed to $a(s)$ since it would be in a higher n_1-state contradicting our choice of s_0. If there is no such s we claim that $R_{\sigma_1} \cap \bar{A}$ is recursive: To see if $x > \Lambda_{n_1}^\omega$ is in $R_{\sigma_1} \cap \bar{A}$ ask first if $x \in R_{\sigma_1}$. If not of course the answer is no. If $x \in R_{\sigma_1}$ wait until $x \in R_{i,s}^{h(i)}$ for each $i \leqq i_0$ for which R_i is infinite. If $x \notin A_s$ then $x \notin A$ by assumption. So if no such s exists $R_{\sigma_1} \cap \bar{A} = R_{i_1}$ for some $i_1 > i_0$. But then each possible value for $h(i_1)$ contradicts one of the clauses of the definition of a preference function for A. □

It is immediate from this lemma that $B \subset_r A$. To see that $B \subset_m A$ suppose $W_e \cup A =^* \mathbf{N}$. We reduce W_e and A to get a recursive R_k with $R_k \subseteq^* W_e$ and $\bar{R}_k \subseteq^* A$, so that $\bar{A} \subseteq^* R_k$. Thus $h(k) = 1$ and again by Lemma 1.4 $A - B \subseteq^* R_k \subseteq W_e$ so $W_e \cup B =^* \mathbf{N}$ as required. □

As a simple application of this theorem (which will also be needed in §3) we show that the property of having an rm subset is well defined on the equivalence classes determined by the relation "major in". We say that $A \equiv_m B$ if $A \cap B \subset_m A \cup B$.

Proposition 1.5. *If $B \subset_m A$ then A has an rm subset iff B does.*

Proof. Suppose h is a Δ_3 preference function for A. We claim it is also one for B:

(1) $(\forall \sigma \subseteq h)(R_\sigma \cap \bar{A}$ is infinite) so $B \subseteq A$ simplies that $(\forall \sigma \subseteq h)(R_\sigma \cap \bar{B}$ is infinite).

(2) Consider any $\sigma \subseteq h$ such that $R_\sigma \cap A$ is infinite. If $R_\sigma \cap B$ were finite then $R_\sigma \cap A \subseteq^* A - B$ and so there would be an infinite recursive subset of $A - B$. Its complement contradicts the definition of $B \subset_m A$.

On the other hand if h is a Δ_3 preference function for B it is also one for A:

(1) For any $\sigma \subseteq h$ $R_\sigma \cap \bar{B}$ is infinite. If $R_\sigma \cap \bar{A}$ were finite then $\bar{R}_\sigma$ violates the definition of $B \subset_m A$.

(2) Immediate as $B \subseteq A$. □

To conclude this section we want to consider the connection between rm subsets and another type of major subset (small major subset) which was introduced by Lachlan [2, theorem 3] as a key element in his decision procedure for the $\forall\exists$-theory of $\mathscr{E}^*$. These sets were named by M. Stob who discovered [19] several of their properties as we note below. Small sets also play an important role in the study of d-simple sets [7]. (Along the way we prove some facts about rm subsets that are needed in [8] for the extended decision procedure.)

DEFINITION 1.6. B is small in A $(B \subset_s A)$ if $(\forall U, V)(V \supseteq U \cap (A - B) \rightarrow (U - A) \cup V$ is r.e.). Of course $B \subset_{sm} A$ or $B \subset_{srm} A$ mean that B is small in A as well as $B \subset_m A$ and $B \subset_{rm} A$.

Now the notions of sm subsets and rm subset are quite different. In some rough sense $B \subset_{rm} A$ means that B is close to A or rather that $A - B$ is thin. Thus for example $\subset_{rm}$ is not transitive. On the other hand, $B \subset_{sm} A$ implies that in some ways B is far away from A. Thus for example Stob notes that if $C \subset B \subset_s A$ then $C \subset_s A$ and similarly if $C \subset_s B \subset A$ then $C \subset_s A$. Indeed no reasonable notion of a major subset B being close to A can force $B \subset_s A$ by another result of Stob: $(\forall B \subset_m A)$ $(\exists C)$ $(B \subset_m C \subset_m A$ & $C \not\subset_s A$ & $B \not\subset_s C)$. These remarks easily combine to show that there are often rm subsets which are not small and sm ones which are not r-maximal. The interesting question is thus when, if ever, can these notions coincide. We have a neat answer only when A is simple.

PROPOSITION 1.7. *If A is simple then A has an* srm *subset iff A is r-maximal.*

PROOF. Lachlan shows [2, theorem 3] that every set has an sm subset. However, as we have noted, if A is r-maximal every major subset is automatically rm. Thus every sm subset of A is srm. On the other hand, suppose $B \subset_{srm} A$ but R splits $\bar{A}$. As $B \subset_r A$ we can assume that $A - B \subsetneq^* R$. (Otherwise use $\bar{R}$.) Putting $U = \bar{R}$ and $V = \emptyset$ in the definition of $B \subset_s A$ we see that $\bar{R} - A$ is r.e. As it is also infinite by our choice of R it violates the simplicity of A for our contradiction. □

Unfortunately this approach cannot be used to completely characterize those sets with srm subsets. Although the same proof shows that if A is simple in any W (i.e. $(\forall V)$ $(W \cap V$ infinite $\rightarrow V \cap A$ infinite)) and $B \subset_{srm} A$ then $W - A$ is r-cohesive, there are nowhere simple sets (i.e., not simple in any W) with srm subsets. To see this we first prove a fact also needed for the decision procedure.

LEMMA 1.8. *If $B \subset_{rm} A$ $(B \subset_{sm} A)$, $A_0 \cup A_1 = A$, $A_0 \cap A_1 = \emptyset$ and $A_i - B$ is infinite (as it must be for $i = 0$ or 1) then $B \cap A_i \subset_{rm} A_i$ $(B \cap A_i \subset_{sm} A_i)$.*

PROOF. (1) $B \cap A_i \subset_m A_i$: If $W \cup A_i =^* \mathbf{N}$ then $W \cup A =^* \mathbf{N}$ so that $W \cup B =^* \mathbf{N}$. So $\bar{A}_i \subseteq^* W$ and $\bar{B} \subseteq^* W$ i.e., $\bar{B} \cup \bar{A}_i \subseteq^* W$. Thus $W \cup (B \cap A_i) =^* \mathbf{N}$.

(2) $B \cap A_i \subset_r A_i$: $A_i - (B \cap A_i) \subseteq A - B$ which is already r-cohesive.

(3) $B \cap A_i \subset_s A_i$: say $V \supseteq U \cap (A_i - (B \cap A_i))$. Let $V' = V \cup (U \cap (A - A_i))$. Since A_0 and A_1 split A V' is r.e. and moreover $V' \supseteq U \cap (A - B)$. As $B \subset_s A$, $(U - A) \cup V'$ is r.e. Rewriting this we have $(U - A) \cup$

$V \cup (U \cap (A - A_i)) = (U - A) \cup (U \cap (A - A_i)) \cup V = (U - A_i) \cup V$ is r.e. as required. Thus $B \cap A_i \subset_s A_i$. □

COROLLARY 1.9. *There is a nowhere simple set whith an* srm *subset.*

PROOF. Choose any $B \subset_{srm} A$. By [16, theorem 2] we can split A into two nowhere simple sets A_0 and A_1. By the lemma then $B \cap A_i \subset_{srm} A_i$ for $i = 0$ or 1. □

As our last remark relative to the general decision procedure problem we note that rm subsets are the only way to get d.r.e. sets (i.e., of the form $A - B$) which are r-cohesive but not co-r.e.

PROPOSITION 1.10. *If $A - B$ is r-cohesive but not co-r.e., then $B \subset_{rm} A$.*

PROOF. Suppose W shows that $B \not\subset_m A$. As $A - B$ is not co-r.e. W must split $A - B$. (Otherwise $W \cup B =^* \overline{A - B}$.) We reduce W and A to get a recursive set $R \subseteq W$ with $\bar{R} \subseteq A$ but then R also splits $A - B$ for our contradiction. □

2. Automorphisms and orbits of atomless hhs sets

Our goal in this section is to construct an atomless hhs set with an rm subset and one without one. This, of course, implies that they are not automorphic. We will also see that the atomless hhs sets fall into infinitely many orbits with respect to the class of Δ_3 automorphisms. Our starting point is the construction of various representations $\mathcal{B}^i = \{b_l^i\}_{l<\omega}$ of the atomless Boolean algebra which are not Δ_3 automorphic of which only some have Δ_3 preference functions h, i.e. $(\forall \sigma \subseteq h)(b_\sigma \neq 0)$. (Recall that as in Definition 1.1 we have defined $b_\sigma = \bigcap_{i \leqq \mathcal{L}\text{th}(\sigma)} b_i^{\sigma(i)}$.) We then show that these properties are preserved by Lachlan's embedding of $\mathcal{B}^i$ onto $\mathcal{L}^*(A^i)$ for various hhs A^i.

As a preliminary step we consider the Boolean algebra $\mathcal{P}^* = \{p_i\}_{i<\omega}$ of sets with primitive recursive characteristic functions p_i modulo finite sets. (As the notation suggests we identify sets and their characteristic functions p_i.) $\mathcal{P}^*$ is obviously atomless. We will also need a couple of other facts about it.

PROPOSITION 2.1. *There is no* $\{0, 1\}$ *valued* Δ_2 *function h such that for every i,*

$$h(i) = 1 \Rightarrow p_i \neq^* \emptyset, \qquad \textit{and}$$

$$h(i) = 0 \Rightarrow \bar{p}_i \neq^* \emptyset.$$

PROOF. Consider any $\{0, 1\}$ valued Δ_2 function h. We can write out the definition of h as $h(i) = 1 \Leftrightarrow \forall x\, \exists y\, T_1(i, x, y)$ and $h(i) = 0 \Leftrightarrow \forall x\, \exists y\, T_0(i, x, y)$

where T_0 and T_1 are primitive recursive relations. Next note that there is a primitive recursive function f such that

$$p_{f(e)}(s)=\begin{cases}0 & \text{if } (\mu x\leqq s)[(\forall y\leqq s)\neg T_0(e,x,y)] \\ & \qquad\leqq(\mu x\leqq s)[(\forall y\leqq s)\neg T_1(e,x,y)] \\ 1 & \text{otherwise.}\end{cases}$$

As $g(e,s)=P_{f(e)}(s)$ is also primitive recursive the recursion theorem for primitive functions gives us an i such that $p_i=p_{f(i)}$. We claim that h fails to have the required property at i:

If $h(i)=1$ then there is an x_0 such that $(\forall y)\neg T_0(i,x_0,y)$. On the other hand $(\forall x\leqq x_0)(\exists y)\,T_1(i,x,y)$. Let s_0 be a bound on these witnesses y for $x\leqq x_0$. For $s\geqq s_0$ we have that

$$(\mu x\leqq s)[(\forall y\leqq s_0)\neg T_0(i,x,y)]\leqq x_0\leqq(\mu x\leqq s_0)[(\forall y\leqq s)\neg T_1(i,x,y)].$$

Thus $p_{f(i)}(s)=0=p_i(s)$ for $s\geqq s_0$ so $p_i=^*\varnothing$ for a contradiction. The argument for $h(i)=0$ is essentially the same. $\square$[†]

PROPOSITION 2.2. *$\leqq_{\mathcal{P}^*}$, i.e., $\subseteq^*$ on $\mathcal{P}^*=\{p_i\}$, is a complete Σ_2 predicate.*

PROOF. First note that $p\subseteq^* p_i\Leftrightarrow(\exists x)(\forall y>x)\ [p_i(y)=1\to p_j(y)=1]$ so that $\leqq_{\mathcal{P}^*}$ is Σ_2. (It is for this reason that we followed a suggestion of C. G. Jockusch and used $\mathcal{P}^*$ instead of the recursive sets $\mathcal{R}$ because $\subset_{\mathcal{R}^*}$ is not Σ_2 although it is of course Δ_3.)

On the other hand we show that the complete Σ_2 set $\mathrm{Fin}=\{e\mid W_e=^*\varnothing\}$ is m-reducible to the relation $\leqq_{\mathcal{P}^*}$, namely $W_e=^*\varnothing$ iff $(\exists x)(\forall\langle y_0,y_1\rangle>x)\neg T(e,y_0,y_1)$, where T is the usual primitive recursive Kleene T-predicate. There is then a recursive f such that

$$p_{f(e)}(\langle y_0,y_1\rangle)=1\Leftrightarrow T(e,y_0,y_1)\qquad\text{and}$$

$$p_{f(e)}(\langle y_0,y_1\rangle)=0\Leftrightarrow\neg T(e,y_0,y_1).$$

Thus $W_e=^*\varnothing\Leftrightarrow p_{f(e)}=^*\varnothing\Leftrightarrow p_{f(e)}\subseteq^*\varnothing$. $\square$[†]

We can now describe the various representations of the atomless Boolean algebra that we need via relativizations of $\mathcal{P}^*$. Let $\mathcal{P}^{B^*}=\{p_i^B\}$ be the sets with characteristic functions p_i^B primitive recursive in B modulo finite sets.

[†] C. G. Jockusch has pointed out that 2.1 and 2.2 can also be derived from a proof that $\{i\mid p_i=^*0\}$ and $\{i\mid \bar{p}_i=^*0\}$ are recursively inseparable relative to $\varnothing'$.

COROLLARY 2.3. *There is no* $\{0, 1\}$ *valued* Δ_3 *function h such that for every i* $h(i) = 1$ *implies* $p_i^K \neq^* \emptyset$ *and* $h(i) = 0$ *implies* $\overline{p_i^K} \neq^* \emptyset$ *where K is the complete r.e. set.*

PROOF. Relativize Proposition 2.1 to K and note that $\Delta_3 = \Delta_2^K$. □

COROLLARY 2.4. *If* $C \in \Sigma_3$ *and* $\emptyset'' \leqq_T C$ *then there is a presentation of the atomless Boolean algebra with its inclusion relation of the same degree as C.*

PROOF. By Sacks' jump theorem [12, ch. 13, XXV] there is an r.e. B such that $B'' \equiv_T C$. The relativization of Proposition 2.2 then says that $\leqq_{\mathscr{P}^B}{}^*$ is complete Σ_2 in B and so of the same degree as C. □

We will now sketch enough of Lachlan's construction [1, theorem 6] of an hhs A with $\mathscr{L}^*(A)$ a perscribed Σ_3 Boolean algebra $\mathscr{B}$ to see that there is in fact a Δ_3 isomorphism between the given presentation of $\mathscr{B}$ and the standard one of $\mathscr{L}^*(A)$ as $\{W_i \cup A\}_{i<\omega}$. On this basis we can go from the algebras presented above to hhs sets whose lattices of supersets have the desired Δ_3 properties.

Lachlan begins with a list $\{b_i\}_{i<\omega}$ of generators for the given Boolean algebra $\mathscr{B}$ and an *associate* of $\mathscr{B}$ which is a map $F_{\mathscr{B}} : 2^{<\omega} \to \{0, 1\}$ such that $F_{\mathscr{B}}(\sigma) = 0$ iff $b_\sigma = 0$. The picture here is of a binary tree with the branching at level i representing intersection with b_i and $\bar{b}_i$ respectively. Thus $F_{\mathscr{B}}$ tells us which Boolean combinations of generators are 0 and so nicely codes the algebra $\mathscr{B}$. Thus for example F is Σ_3, i.e., $F_{\mathscr{B}}(\sigma) = 0$ is a Σ_3 predicate, iff $\leqq_{\mathscr{B}}$ is Σ_3. Indeed it is easy to see that $F_{\mathscr{B}}$ and $\leqq_{\mathscr{B}}$ are of the same Turing degree.

The next step is to build the desired hhs set A. Lachlan's construction may be pictured as beginning with a binary tree. Steps in the construction consist of putting numbers (eventually all of them) on nodes of the tree and moving them from one node to another or off the tree entirely which corresponds to putting them into A. Numbers are moved subject to various geometric constraints as to the type of motion allowed so as to maximize the e-states of occupants of nodes of level e. The motion is also of course guided by an approximation to the Σ_3 associate F so as to guarantee the $\mathscr{L}^*(A)$ is in fact isomorphic to $\mathscr{B}$.

If we let C_σ be the elements that from some stage of the construction onward rest on some node $\tau \supseteq \sigma$, then we can state the main achievements of the constraints on, and requirements of, the construction as follows:

(1) $(\forall\sigma)[A \cup C_\sigma$ is r.e.$]$.

(2) $(\forall\sigma)[C_\sigma =^* \emptyset \Leftrightarrow F(\sigma) = 0]$.

(3) $(\forall i)(\forall\sigma)_{lth\sigma=i+1}[C_\sigma \subseteq^* W_i \vee C_\sigma \cap W_i =^* \emptyset]$.

It is clear from these properties that if we let $\{a_i = A \cup \bigcup\{c_{\sigma^\frown 1} \mid lth\sigma = i\}\}_{i<\omega}$ be a set of generators for a Boolean algebra of sets $\mathscr{A}$ then $F_{\mathscr{A}} = F_{\mathscr{B}}$ so that $\mathscr{A} \cong \mathscr{B}$ via the map sending a_i to b_i. Now it is also easy to see that not only is $\mathscr{A} \cong \mathscr{L}^*(A)$ but in fact there is a Δ_3 isomorphism between the standard presentation of $\mathscr{L}^*(A)$ and $\mathscr{A}$ given by $\varphi(W_i \cup A) = \bigcup\{a_\sigma \mid \mathrm{lth}\sigma = i+1\ \&\ C_\sigma \subseteq^* W_i\}$. That $\varphi(W_i \cup A) =^* W_i \cup A$ is immediate from (3). To see that φ is Δ_3 it suffices to show that $C_\sigma \subseteq^* W_i$ is Δ_3. First

$$C_\sigma \subseteq^* W_i \Leftrightarrow (\exists x)(\forall y > x)[y \in C_\sigma \rightarrow y \in W_i]$$

$$\Leftrightarrow (\exists x)(\forall y > x)[(\exists \tau \supseteq \sigma)(\exists s)(\forall t > s)$$

$$[y \text{ is on node } \tau \text{ at stage } t \text{ of the construction}] \rightarrow y \in W_i].$$

Thus $C_\sigma \subseteq^* W_i$ is Σ_3. On the other hand by (3) $C_\sigma \subseteq^* W_i$ iff $W_i \cap C_\sigma$ is infinite iff $(\forall x)(\exists y > x)[y \in W_i\ \&\ y \in C_\sigma]$ which by the definition of C_σ is Π_3. Thus φ is in fact Δ_3. Composing maps we now have a Δ_3 isomorphism ψ between the standard presentation of $\mathscr{L}^*(A)$ and the given one of $\mathscr{B}$. This enables us to show that $\mathscr{L}^*(A)$ inherits the desired properties from the Boolean algebras discussed at the beginning of this section.

THEOREM 2.5. *For every $C \in \Sigma_3$ with $\varnothing'' \leqq_T C$ there is an atomless hhs A with $\leqq_{\mathscr{L}^*(A)} \equiv_T C$.*

PROOF. Choose B as in Corollary 2.4 with $B'' \equiv_T C$ and let $\mathscr{B} = \mathscr{P}^{B^*}$ so that $\leqq_{\mathscr{B}} \equiv_T C$. By the above discussion we have a Δ_3, and so recursive in $\varnothing''$, isomorphism $\psi : \mathscr{L}^*(A) \rightarrow \mathscr{B}$. As $\varnothing'' \leqq_T C$ it suffices for this theorem to show that $\varnothing'' \leqq_T \leqq_{\mathscr{L}^*(A)}$ as well: Let f be a recursive function such that $W_{f(p)} = \{x \mid (\exists y > x)[y \in W_e]\}$. Then W_e is finite iff $W_{f(p)} \cup A \subseteq^* A$. □

COROLLARY 2.6. *There are infinitely many atomless hhs sets A_i such that for any $i \neq j$ $\mathscr{L}^*(A_i)$ is not Δ_3 isomorphic to $\mathscr{L}^*(A_j)$.*

PROOF. As $\varnothing'' \leqq_T \mathscr{L}^*(A_i)$ for every i any Δ_3 isomorphism would preserve the Turing degree of $\leqq_{\mathscr{L}^*(A)}$. It therefore suffices to choose the A_i given by the theorem for C's of different degree. □

Note that these methods cannot be applied to non-hhs sets since if A is not hhs then $\leqq_{\mathscr{L}^*(A)}$ is Σ_3 complete: Let $W_{f(x)}$ be a weak array showing that A is not hhs. To see that $\{e \mid W_e \text{ is cofinite}\}$, a Σ_3 complete set, is m-reducible to $\leqq_{\mathscr{L}^*(A)}$ just note that W_e is cofinite $\Leftrightarrow \bigcup_{x<\omega} W_{f(x)} \cup A \subseteq^* \bigcup_{x \in W_e} W_{f(x)} \cup A$.

THEOREM 2.7. *There is an atomless hhs set A with no Δ_3 preference function and hence no rm subset.*

PROOF. Let A be the set constructed as above beginning with the Boolean algebra $\mathcal{P}^{K^*}$ and ψ be the associated Δ_3 isomorphism. As ψ^{-1} is also Δ_3 we can clearly find a Δ_3 function g such that $\psi(W_{g(i)} \cup A) = p_i^K$. As $\mathcal{L}^*(A)$ is a Boolean algebra we know that for every W_i there is a W_j such that $W_i \cup W_j \cup A = \mathbf{N}$ and $W_i \cap W_j \subseteq A$. As both of these conditions are recursive in $\varnothing''$ (the first is Π_2, the second Π_1) we can get a recursive in $\varnothing''$ and so Δ_3 function f such that $f(i)$ is the desired W_j. We can then reduce $W_i \cup A$ and $W_{f(i)}$ to get a recursive set $R \subseteq W_i \cup A$ with $\bar{R} \subseteq W_{f(i)}$. As an r.e. index for R can be found effectively from i and $f(i)$ we have a Δ_3 function d such that $R_{d(i)} \subseteq W_i \cup A$ and $\bar{R}_{d(i)} \subseteq W_{f(i)}$. It is now easy to see that if h were a Δ_3 preference function for A then the Δ_3 function hdg would violate Corollary 2.3. Thus for example $hdg(i) = 1 \Rightarrow R_{dg(i)} \cap \bar{A}$ is infinite $\Rightarrow W_{g(i)} \cap \bar{A}$ is infinite $\Rightarrow W_{g(i)} \cup A \neq^* A \Rightarrow p_i^K \neq^* \varnothing$. □

THEOREM 2.8. *There is an atomless hhs set A with a Δ_3 preference function and so an* rm *subset.*

PROOF. Let $\mathcal{B} = \{b_i\}_{i<\omega}$ be a recursive presentation of the atomless Boolean algebra and A the hhs set constructed from it as above. Again $\psi : \mathcal{L}^*(A) \to \mathcal{B}$ is a Δ_3 isomorphism. Now $W_i \cap \bar{A}$ is infinite iff $\psi(W_i \cup A) \not\leq_{\mathcal{B}} 0$. As this last relation is recursive $\{i \mid W_i \cap \bar{A}$ is infinite$\}$ is Δ_3 and so recursive in $\varnothing''$ as is $\{i \mid R_i \cap \bar{A}$ is infinite$\}$. We now define our preference function h recursively in $\varnothing''$ by induction. $h(0) = 1$ iff $R_0 \cap \bar{A}$ is infinite. Let σ be h restricted to i. Now $h(i) = 1$ iff $R_\sigma \cap R_i \cap \bar{A}$ is infinite. We can find an index for R_σ recursively in $\varnothing''$ so h is recursive in $\varnothing''$. As $R_\sigma \cap \bar{A}$ is infinite by induction if $R_\sigma \cap R_i \cap \bar{A}$ is not then $R_{\sigma * h(i)} \cap \bar{A} = R_\sigma \cap \bar{R}_i \cap \bar{A}$ is infinite as required. □

Our main result is an immediate consequence of these last two theorems.

THEOREM 2.9. *There are two atomless hhs sets which are not automorphic.* □

3. Degree classes

In this section we will consider the question of which degrees contain sets with or without rm subsets. (Recall that we consider only r.e. degrees.) As the property of having an rm subset is inherited downward on simple sets it is easy to see that such sets exist in every non-recursive degree. On the other hand we will diagonalize over Δ_3 functions to produce in each non-recursive degree a set with no rm subset. Here however non-simplicity plays a necessary role as every simple set A with $\deg A \in \mathbf{L}_2$ (i.e. $A'' \equiv_T \varnothing''$) has an rm subset. As far as the degree question for simple sets with no rm subsets is concerned we can only show that a familiar situation obtains. The class of degrees is trapped between $\bar{\mathbf{L}}_2$

and $\boldsymbol{H}_1$ (the r.e. degrees with jump $\boldsymbol{0}''$). We first examine the simple sets with rm subsets.

THEOREM 3.1. *Every simple set has simple subsets of each non-recursive degree. Thus in every non-recursive degree there are simple sets with* rm *subsets.*

PROOF. Let M be any simple set. By Robinson [11, theorem 2] we can choose a $B \in \boldsymbol{a}$ with an enumeration function b such that the associated computation function fails to dominate some recursive f. We can then easily choose an enumeration m of M so that the associated computation function does dominate f. If we then let A be the set of elements of M permitted by B, $\{m(s) \mid (\exists t > s)(b(t) \leqq m(s))\}$, A is our required set. As usual $A \leqq_T B$ while the choice of enumerations guarantees that $M - A$ is infinite and so $B \leqq_T A$. If W_e is infinite, $W_e \cap M$ is infinite. So if $W_e \cap A$ were finite we could enumerate an infinite list of elements in $M - A$ and so compute B effectively. Thus A is simple. As having a Δ_3 preference function is inherited downward on simple sets (1.1a) if we choose M to have an rm subset A will also have one. □

Now we turn to non-simple sets without rm subsets.

THEOREM 3.2. *Every* $\boldsymbol{a} > 0$ *contains a set* A *with no* rm *subset.*

PROOF. We will build A while diagonalizing over all Δ_3 preference functions to guarantee that A has none and so no rm subset. Let $B \in \boldsymbol{a}$. Our positive requirements will try to code B into A in a nice way. To spread them out and minimize interference with the negative requirements we first choose a recursive sequence $\{Q_i\}$ of recursive sets such that $\forall \sigma$ (Q_σ is infinite) and $\bigcup_{i<\omega} Q_i = \mathbf{N}$. We then choose a strong array of disjoint finite sets $\{C_e\}_{e<\omega}$ with $\bigcup_{e<\omega} C_e = \mathbf{N}$ such that $(\forall \sigma)_{\mathrm{lth}\,\sigma < e}[C_e \cap Q_\sigma \neq \varnothing]$. Our positive requirements are then given by

$$P_e\text{: If } e \in B \text{ put one element of } C_e \text{ into } A.$$

It is clear that if the P_e all succeed and that they are the only causes of elements entering A then $B \equiv_T A$. (First $e \in B$ iff $C_e \cap A \neq \varnothing$. Next to see whether $x \in A$ find an e such that $x \in C_e$. If $e \in B$ wait until P_e succeeds and see if x is put into A. If $e \notin B$, $x \notin A$.)

The negative requirements are designed to guarantee that there is no Δ_3 preference function for A. We will list all possible $\{0, 1\}$ valued Δ_3 functions by listing all pairs $\langle S_i, T_i \rangle$ of Σ_3 predicates. The intention is that for each such function h there is an i such that $\varphi(x) = 1 \Leftrightarrow S_i(x)$ and $\varphi(x) = 0 \Leftrightarrow T_i(x)$. For the diagonalization we choose a recursive function f such that $Q_i = R_{f(i)}$. Our requirement for each i is then

N_i: If $\langle S_i, T_i\rangle$ define a Δ_3 function h as above then $Q_i^{h(f(i))} \cap \bar{A} = R_{f(i)}^{h(f(i))} \cap \bar{A}$ is recursive.

Now if h is defined by $\langle S_i, T_i\rangle$ and N_i succeeds then h cannot be a preference function for A : $R_{f(i)}^{h(f(i))} \cap \bar{A} = R_k$ for some $k \geqq f(i)$ by assumption. Any value for $h(k)$ will thus contradict the definition of a preference function as $R_k \cap A = \varnothing$ and $R_{f(i)}^{h(f(i))} \cap \bar{R}_k \cap \bar{A} = \varnothing$. Thus if all the N_i succeed A will have no rm subset.

To approximate the Σ_3 predicates S_i and T_i in the construction [12, p. 326] we give simultaneously r.e. sequences of r.e. sets S_i^n and T_i^n so that

$$S_i(f(i)) \Leftrightarrow (\exists n)[S_i^n \text{ is infinite}] \qquad \text{and}$$

$$T_i(g(i)) \Leftrightarrow (\exists n)[T_i^n \text{ is infinite}].$$

In the construction we proceed on the assumption that the value of the associated function is that associated with the list which first has an infinite set on it. (Of course if $\langle S_i, T_i\rangle$ correctly defines some h then only one list has an infinite set anyway.) In this way we will guess which of $Q_i \cap \bar{A}$, $\bar{Q}_i \cap \bar{A}$ we wish to make recursive by keeping elements out of A. The ordering of priorities for fixed i is $S_i^n < T_i^m$ iff $n \leqq m$. To be precise we describe the actual construction.

CONSTRUCTION. Stage s: Say $e = b(s)$ where b enumerates B. Define a σ of length e by $\sigma(i) = 0$ iff

$$(\mu n)(\exists x)[x \in C_e \cap Q_i \text{ and } x \leqq \max(S_{i,s}^n)]$$

$$\leqq (\mu n)(\exists x)[x \in C_e \cap \bar{Q}_i \text{ and } x \leqq \max(T_{i,s}^n)].$$

(Otherwise $\sigma(i) = 1$.) We now put an element of $C_e \cap Q_\sigma$ into A.

The positive requirements are obviously satisfied. To establish the theorem we need only show that if $\langle S_i, T_i\rangle$ defines a Δ_3 function h then $Q_i^{h(f(i))} \cap \bar{A}$ is recursive. Fix i and suppose that $h(f(i)) = 1$ (the proof is similar if $h(i) = 0$). Let n be the least n such that S_i^n is infinite. (Of course for no m is T_i^m infinite by assumption.) Let $m = \max(\bigcup_{j \leq n} T_i^j)$. To see if an $x > m + \max\{y \mid (\exists e)[y \in C_e$ & $(\exists z < m)[(z \in \bar{C}_e]]\}$ which is in Q_i is also in $\bar{A}$ find an s such that $\max S_{i,s}^n > x$. We claim that if $x \notin A_s$ then $x \notin A$. Suppose $x \in C_e$ and $e = b(t)$, $t \geqq s$. (Otherwise x has no chance of being put into $A - A_s$.) Thus $x \in C_e \cap Q_i$ and $x \leqq \max S_{i,t}^n$ but no element of C_e is less than m so there is no element of $C_e \cap \bar{Q}_i$ less than $\max T_{i,t}^j$ for $j \leqq n$. Thus at stage t $\sigma(i) = 0$ and we put an element of $\bar{Q}_i$ into A. Of course x never again has a chance to be put into A. □

Note that the negative requirements to make a set nowhere simple as in [16] can easily be combined with this construction just by making the C_e larger (e.g. $|Q_\sigma \cap C_e| > e$) to produce a nowhere simple set with no rm subset. Combining this with Corollary 1.9 we see that at the opposite end of the scale from hhs sets one gets nowhere simple sets with and without rm subsets.

Our final results will deal with the class of degrees $\boldsymbol{F} = \{\boldsymbol{a} \mid$ there is a simple $A \in \boldsymbol{a}$ with no rm subset$\}$. We will show that $\bar{\boldsymbol{L}}_2 \supseteq \boldsymbol{F} \supseteq \boldsymbol{H}_1$.

THEOREM 3.3. *If A is simple and $A \in L_2$ then A has an* rm *subset.*

PROOF. As $A \in L_2$, i.e., $A'' \equiv_T \varnothing''$ it is immediate that $\{e \mid W_e \cap \bar{A}$ infinite$\} \leqq_T \varnothing''$. As we have already seen in the proof of Theorem 2.8 this suffices to construct a Δ_3 preference function for A. □

THEOREM 3.4. *If $\boldsymbol{b} \in \boldsymbol{H}_1$ there is a simple set $B \in \boldsymbol{b}$ with no* rm *subset.*

PROOF. Let A be the hhs set with no rm subset constructed in Theorem 2.7. By Lerman [5] we may take $B \subset_m A$ of degree $\boldsymbol{b}$. By Proposition 1.5 B has no rm subset. Moreover B is simple: If W is infinite but $W \cap B = \varnothing$, then as A is simple $\varnothing \neq^* W \cap A \subseteq A - B$. There is then an infinite recursive $R \subseteq W \cap A \subseteq A - B$ and $\bar{R}$ violates the majority of B in A. □

4. Final remarks and open questions

In Section 2 we refuted the conjecture that $\mathcal{L}^*(A) \cong \mathcal{L}^*(B)$ implies A automorphic to B for all hhs sets A, B. Is this true for any hhs set A beside those where $\mathcal{L}^*(A)$ is a finite Boolean algebra as proved in [17, corollary 2.6]? Are there any other classes of r.e. sets besides hhs sets where the conjecture holds? The Post program [10] of classifying r.e. sets A in terms of the lattice of supersets $\mathcal{L}^*(A)$ is seen to be increasingly inadequate for determining the atutomorphism type of A. Rather one must examine relations between an r.e. set A and its complement. For example, if A is simple then having a Δ_3 preference function is a property solely of $\bar{A}$ and yet determines whether A can possess certain kinds of subsets. Other examples of properties relating A to $\bar{A}$ are the d-simple sets [7], the extension theorem for generating automorphisms [17, theorem 2.2], and the small major subsets [2, theorem 3] of the Lachlan decision procedure. There are undoubtedly many more of these properties and they will play a crucial role in the final classification of automorphism types of r.e. sets and in the decision procedure for its elementary theory.

In Corollary 2.6 we produce atomless hhs sets $\{A_i\}_{i<\omega}$ such that for $i \neq j$, $\mathcal{L}^*(A_i)$ is not Δ_3 isomorphic to $\mathcal{L}^*(A_j)$, and hence A_i is not automorphic to A_j

by the known automorphism methods [17] and [18] which always generate Δ_3 automorphisms. As a final attempt to generalize the maximal set automorphism result [17] to hhs sets, suppose that $\mathcal{L}^*(A)$ and $\mathcal{L}^*(B)$ are isomorphic by a Δ_3 isomorphism. Is A necessarily automorphic to B? Which classes of r.e. sets besides maximal sets and infinite, coinfinite recursive sets constitute orbits (i.e., any two members are automorphic)?

Finally, one of the most interesting questions on r.e. sets and their degrees is the classification of degrees of classes of r.e. sets which are invariant under automorphisms. Martin [9] showed that $\boldsymbol{H}_1 = \{\boldsymbol{d} : \boldsymbol{d}$ contains a maximal set$\}$, and Lachlan [3, theorem 4] and Shoenfield [15] showed that $\bar{\boldsymbol{L}}_2 = \{\boldsymbol{d} : \boldsymbol{d}$ contains an atomless coinfinite r.e. set$\}$. Lerman and Soare showed [7] that not every invariant class $\boldsymbol{C}$ of r.e. degrees is of the form $\boldsymbol{H}_n$ or $\bar{\boldsymbol{L}}_n$ since $\boldsymbol{D}$, the degrees containing d-simple sets, satisfy $\boldsymbol{H}_1 \subseteq \boldsymbol{D}$ and $\boldsymbol{D}$ splits $\boldsymbol{L}_1$. In Section 3 we showed that $\boldsymbol{H}_1 \subseteq \boldsymbol{F} \subseteq \bar{\boldsymbol{L}}_2$. What is the exact classification of $\boldsymbol{F}$? Which of the other high and low classes $\boldsymbol{H}_n$ and $\boldsymbol{L}_n$ of r.e. degrees are invariant? Recall that

$$\boldsymbol{H}_n = \{\boldsymbol{d} : \boldsymbol{d} \text{ r.e. and } \boldsymbol{d}^{(n)} = \boldsymbol{0}^{(n+1)}\}, \qquad \text{and}$$

$$\boldsymbol{L}_n = \{\boldsymbol{d} : \boldsymbol{d} \text{ r.e. and } \boldsymbol{s}^{(n)} = \boldsymbol{0}^{(n)}\}.$$

What is the classification of the coinfinite r.e. sets having no r-maximal superset? Which other simple sets having no $r$mm subset exist besides the few atomless hhs sets in Section 2 and their major subsets? There is a direct construction of a simple set with no Δ_3 preference function (and hence no $r$mm subset) obtained by diagonalizing over all possible Δ_3 functions. Variations of this construction may lead to an exact classification of $\boldsymbol{F}$.

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